Scale-free percolation mixing time

Pavia-Milano Seminar series on Probability and Mathematical Statistics $(PMS)^2$

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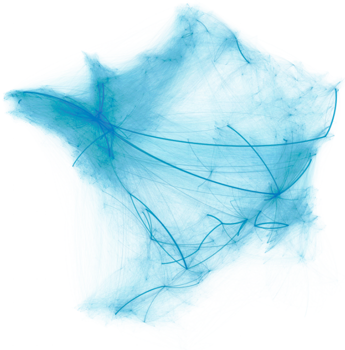
Università degli studi Tor Vergata

May 9th, 2022

Complex networks



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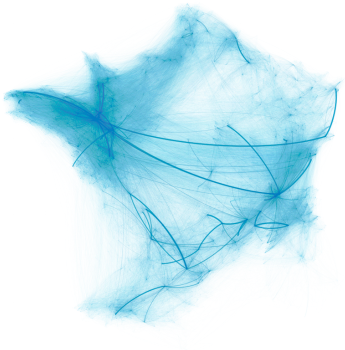

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Law of degrees decays polynomially:

$$\mathbb{P}(D_x \geq t) \simeq t^{-\gamma}$$

Small world

Graph dist $\simeq$ log(Euclidean dist)

Positive clustering coefficient

Probability that two of my friends are friends is high.

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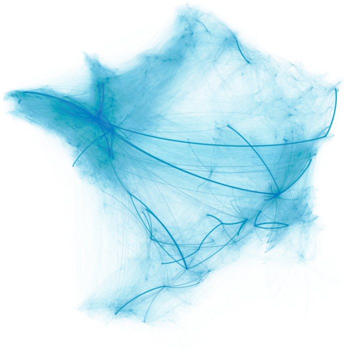

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	Scale-free	Small world	Positive CC
Erdös-Rényi	✗	✓	✗
Norros-Reittu, Chung-Lu	✓	✓	✗
Watts-Strogatz	✗	✓	✓

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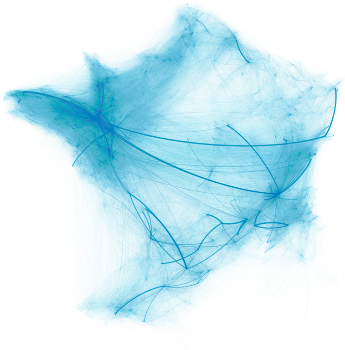

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Inhomogeneous spatial random graphs!

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STEP 1 : For all $x \in V$ sample an "importance" W_x. How?

$$P(W_x \geq w) = w^{-\tau}$$

STEP 2 : For all $x, y \in V$

$$P(x \leftrightarrow y) = 1 - \mathrm{e}^{-\frac{W_x W_y}{\|x-y\|^{\alpha}}} \simeq \frac{W_x W_y}{\|x-y\|^{\alpha}}$$

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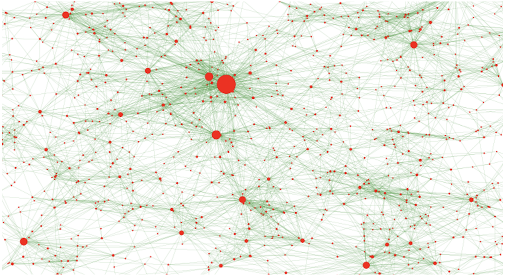

$\tau: \; P(W_x \geq w) = w^{-\tau}$ $\qquad \alpha: \; P(x \leftrightarrow y) \simeq \dfrac{W_x W_y}{\|x-y\|^{\alpha}}$ $\qquad \gamma := \dfrac{\alpha\tau}{d}$

Theorem ([Deijfen&al.'13, Deprez&al.'14, Hao, Heydenreich '21, S., Dalmau '21...])

DEGREE *If* $\alpha > d$ *and* $\gamma > 1$

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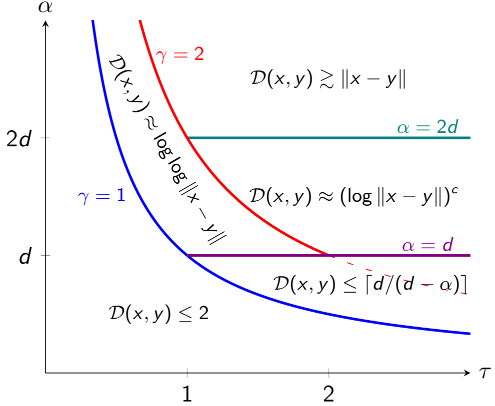
DISTANCES *Let $\mathcal{D}(x,y) :=$ graph distance of x and y.*



Mixing time for the simple random walk

G_N SFP on $V_N = \mathbb{Z}/N\mathbb{Z}$. Simple random walk on G_N

$$P^{G_N}(X_{t+1} = y \,|\, X_t = x) = \frac{1}{D_x} \mathbb{1}_{\{x \leftrightarrow y \text{ in } G_N\}}$$

X_t approaches π as $t \to \infty$, with $\pi(x) = \frac{D_x}{2|E(G_N)|}$. But...

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... how fast?

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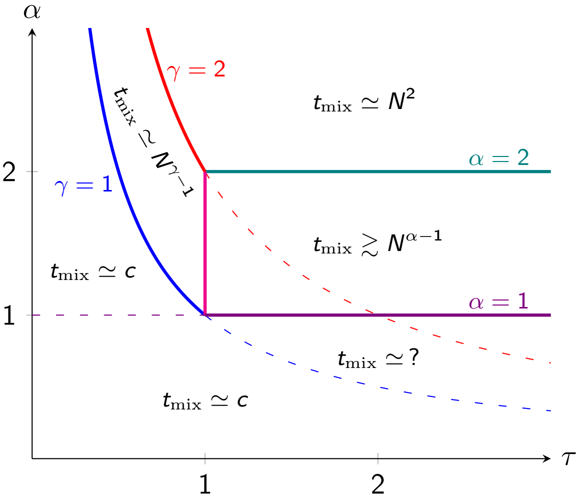
For long-range percolation (no weights, $\tau = \infty$)

$$t_{\text{mix}}(N) \simeq \begin{cases} N^{\alpha-1} & \text{if } 1 < \alpha < 2 \\ N^2 & \text{if } \alpha > 2 \end{cases}$$ [Benjamini&al. '09]

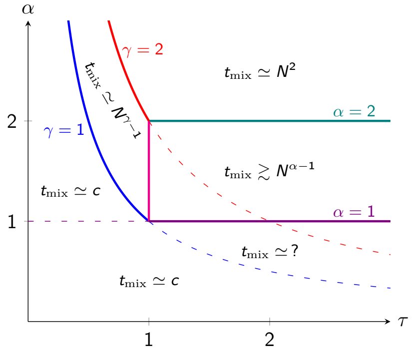
"Small diameter yet large polynomial mixing time"

Main result

Theorem (A. Cipriani, S. '21)



Intuition



α
$\gamma = 2$
$t_{\rm mix} \simeq N^2$
$t_{\rm mix} \gtrsim N^{\gamma-1}$
$\alpha = 2$
2
$\gamma = 1$
$t_{\rm mix} \gtrsim N^{\alpha-1}$
$t_{\rm mix} \simeq c$
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1
$t_{\rm mix} \simeq ?$
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1
2
τ

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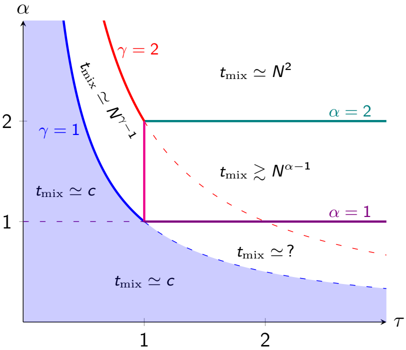

$\gamma < 1$

$\mathbb{E}[D_x] = \infty$ (on $\mathbb{Z}$),
distances bounded by a constant
$\implies$ polylogarithmic mixing

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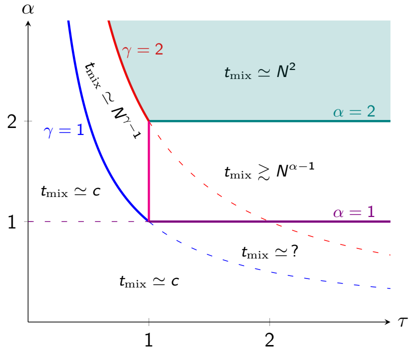

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graph distances like Euclidean distances
$\implies$ slowest mixing

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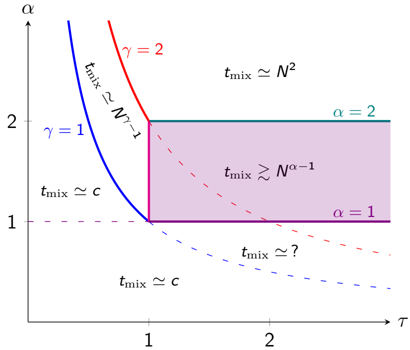

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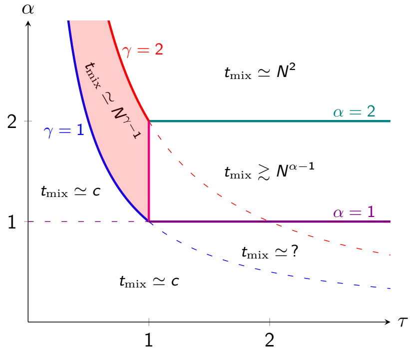

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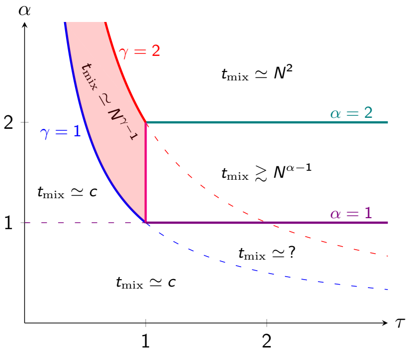
- $1 < \gamma < 2,\ \tau < 1$

  $\mathbb{E}[D_x] < \infty,\ \mathrm{Var}(D_x) = \infty \implies$ Hubs speed up the mixing.

Proof of $1 < \gamma < 2,\ \tau < 1$



Thank you!